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PROJECTIVE AND METRIC GEOMETRY*

BY EDWIN BIDWELL WILSON

THE passage from synthetic projective to metric geometry is seldom made in a systematic and satisfactory manner. Those whose point of view is pure synthetic geometry, for example Reye or Böger or Enriques, assume the metric geometry as given and confine their attention to interpreting it in terms of projective relations. This fails to bring out the logical step made in passing from the one to the other. Those who, like Klein and Cayley, approach the subject from the side of non-euclidean geometry introduce the circular points at infinity which even when not directly dependent on analysis are out of the spirit of pure geometry. In any case the start is made from the number-system defined upon a straight line by means of harmonic constructions. Let us start from this point.

Consider first the straight line. In projective geometry the linear group of transformations of the line into itself contains three parameters, if we may use this convenient analytic term to stand for the geometric conception of degrees of freedom. The establishment, upon the line, of a definite number-system dependent on three assumed points 0, 1, and ∞ determines completely the theory of segments and distance — the difference between two numbers giving the distance between the points to which they are affixed. The postulate of congruence, namely, that from a given point of the line and in either direction there may be laid off upon the line one and only one segment equal to a given segment of the line, has an evident interpretation. It is this postulate of the conservation of distance which in reality introduces the essential differences between projective and metric geometry; for it cuts down the number of parameters in the group to one. In all transformations in which distance remains invariant the point which has been assumed as ∞ remains fixed and the point 1 must be carried into the point $n + 1$ or $n - 1$ if the point 0 is carried into the point n. Whether this definition of distance has any connection or not with the ordinary conceptions connected with distance in real space depends on the further axiom (not postulate) that if the point ∞ be taken at

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what we consider to be an "infinite distance" the congruent segments become identical with our ordinary *equal* segments.

Let us pass to the plane. The fundamental postulate of projective geometry is that every two lines meet in a point. As a first step towards metric geometry this may be restated as follows: The points of the plane may be separated into two classes of which one consists of actual points and the other of ideal points such that there is through a given point one and only one line which shall cut a given line in an ideal point unless the line be composed wholly of ideal points or the given point be itself an ideal point. From this it follows that every line which contains one actual point contains only one ideal point and that a line which contains two ideal points is made up wholly of such points. That there must be an infinity of ideal points is evident on considering a pencil of lines passing through a common point — for each of the lines contains an ideal point and no two can meet in the same point other than the vertex of the pencil. The ensemble of ideal points therefore fills out a line in the projective plane.

In discussing actual plane space it is to be taken as an axiom that through a given point there is one and only one line which does not cut a given line. The procedure of von Staudt is the converse of this and herein lies a slight departure from the true spirit of projective geometry. Von Staudt's problem is to discuss the "Schein" or *look* of space relations. Now, if two lines meet they look as if they met and if they do not meet they none the less look as if they did. Consequently the question of meeting or not meeting has no place in pure projective geometry. The distinction which is first made in passing to metric geometry is the assumption that though any two lines look as if they meet, in reality some lines do not meet. As a consequence of this a further difference between projective and metric geometry may be noted. The projective plane is doubly connected. One line in the plane does not divide the plane into two parts but two lines do separate the plane into two distinct portions. In passing to metric geometry the points on the ideal line are, so to speak, thrown out of the plane leaving a cut in the plane which renders it simply connected so that a line does separate the metric plane into two parts.

The group of transformations which leave invariant the ideal line forms a six-parametered subgroup of the general projective group and possesses the characteristic property of multiplying all areas in the same ratio. There is a further subgroup which not only leaves the ideal line invariant but also the areas of figures in the plane. To discuss these two groups which give rise to geometries less general than the projective and less restricted than the metric is now possible by making use of the ideas developed by the author in connection with the projective definition of area :* but we shall not stop to do so here.

If in setting up a number-system on any line in the plane we choose the ideal point of that line as the infinite point we have upon that line the metric geometry explained in the second paragraph. To pass to the idea of distance in the plane we assume a fixed point O and a conic K subject to the sole condition that O and the ideal line shall be pole and polar with respect to K. If the metric geometry to be obtained is to resemble our ordinary euclidean, the point O and the ideal line must be separated by K — that is, the involution set up on the ideal line by the association of points with the points in which the polars of those points with respect to K cut the ideal line must be an elliptic involution. The conic K is assumed to be a unit conic. By this is meant that the hitherto undefined unit of distance on the various lines issuing from the point O is taken arbitrarily to be the distance intercepted between the point O and the conic K. The transformations that leave K, O, and the ideal line invariant are a singly infinite set which corresponds to the group of rotations in the euclidean plane.

Upon the pencil of lines issuing from O construct number-systems, choosing O and the point where the line cuts the unit conic and the point of intersection of the line with the ideal line as 0, 1 and ∞ respectively. The locus of points which correspond to a given number is a conic having O and the ideal line as pole and polar and defining the same involution on the ideal line as the conic K. To prove this we note that the points $0, 1, x, \infty$ of one line and the points $0, 1, x, \infty$ of any other line of the pencil whose vertex is O are in perspective with respect to a point on the ideal line.. Hence the chords 11 and xx meet on the ideal line and in particular the tangents to the two curves at corresponding points meet on that line. This is precisely the property of the conic which cuts out the same involution on the ideal line as is cut out by K.

To find the distance of two points A and B taken at random in the plane one needs only to join A to O, lay off on the line AO from the point A a

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* See "A Generalized Conception of Area: Applications to Collineations in the Plane," ANNALS OF MATHEMATICS, vol. 5, pp. 29–45, October, 1903. Also "Ueber eine von dem Begriffe der Länge unabhängige Definition des Volumens," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 12, pp. 555-561, December, 1903.

segment of unit length (the number-system on OA suffices to determine this unit), then draw about A with A and the ideal line as pole and polar the conic which has the same involution on the ideal line as K has (this is the unit conic about A), and finally construct upon the line AB a number-system using as 0, 1, ∞ the point A, the intersection of the unit conic about A with AB, and the point where AB cuts the ideal line. The definition of distance is such as to render evident the statement that distance is left invariant by a three-parametered subgroup of the general projective group and of its two subgroups mentioned above. This group corresponds to the group of rigid motions and transformations of symmetry in the Euclidean plane. Figures which may be carried into one another by transformations of this group will be said to be congruent.

The assumption of an ideal line and a conic have enabled us to assign a number, which we called the distance, to every segment in the plane. That these segments follow the ordinary postulates, such as are given in Hilbert's *Festschrift* or Veronese's *Elementi di Geometria*, is too evident to require further comment. As yet we have made no mention of angles. We are however quite in a position to take up the theorem that: Two triangles which have three sides of the one equal in length to three sides of the other are congruent. This theorem is almost invariably made to depend on the concept of congruent angles no matter what system of postulates is laid down. For our proof we need the following proposition of projective geometry: Two conics whose poles with respect to a given line are A and B respectively and which determine the same involution on that line will intersect in no points or will be tangent in some point of the line AB or will intersect in two points harmonically situated with respect to the line AB and the given line. As a corollary of this theorem, the sum of two sides of a triangle is greater than the third side. And conversely if the sum of two segments is greater than a third, two triangles may be constructed which shall have these segments for sides and the third segment for base. The two triangles will be harmonically situated with respect to their base and the given line. Two such triangles may be called symmetrical with respect to their common base. It follows immediately that if we disregard the order of the sides of a triangle two triangles which have their sides respectively equal in length are congruent.

An angle is said to be formed when two lines meet in a point and terminate there. Two angles will be said to be congruent when and only when the triangles formed by joining the points at unit distance from the vertex are congruent. Evidently if instead of taking unit distances any arbitrary segment had been laid off on one side of each angle and any other arbitrary distance on the other side, the triangles thus formed are congruent. Right angles are definable in the usual manner as formed by one line meeting another so that the adjacent angles are congruent. At a point in a line one and only one perpendicular to the line may be erected, the locus of points equally distant from two given points is the perpendicular bisector of the segment joining them, and similar theorems may now be proved. Let two lines meet at right angles in the point Q. Draw about the point the unit conic and suppose it cuts one line in A and C and the other line in B and D. Draw the set of lines parallel to AC, that is, perpendicular to BD. Now, BD bisects the segments intercepted on these lines by the conic. Hence BD is the polar of the point at infinity on the line AC. Hence two perpendicular lines cut the line at infinity, that is, the ideal line, in corresponding points of the involution cut out on that line by the unit conic. This corresponds to the statement usually made that, perpendicular lines are harmonically situated with respect to the circular points at infinity.

From this point on, the development of plane geometry whether carried on analytically or synthetically does not differ from the ordinary. It will have been noticed that we have been able to avoid the noneuclidean absolute by the use of von Staudt's ideas, by replacing the conception of a conic intersecting a line in imaginary points by the conception of the involution set up on the line. Evidently a similar line of reasoning could be followed in space. There would be an ideal plane and a unit quadric surface to be assumed. We should speak not of the " spherical circle at infinity " but of the polarity set up in the ideal plane by associating the points of that plane with the lines in which the polar planes of those points with respect to the quadric cut the plane.

One remark remains. We have assumed a unit conic and have in no way restricted it to being a circle. Have we an euclidean geometry? As far as pure logic goes we have. As concerns actual plane space we have something considerably different. We know that in reality one segment is sufficient to establish a measure of distance throughout space. We have one degree of freedom only in selecting our unit, whereas thus far we have assumed three degrees of freedom in the selection. From any standpoint other than the purely logical this is of importance and serves to illustrate what Poincaré means when he states in his *La science et l'hypothèse* that in addition to that which is usually laid down in our systems of axioms we have certain conceptions far more subtle and no less fundamental, such as the isotropy of space, for example. The metric geometry discussed above is not isotropic unless the unit conic be that which we understand to be a circle and only then does it become our actual euclidean geometry. As an axiom is a self-evident truth we might consistently build up our geometry quite without axioms; but having gone so far as to include the parallel axiom we ought to go on and include the still more evident axiom of isotropy—and so we should indeed, did not its very evidence make it abnormally hard to state.

YALE UNIVERSITY,
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